

Polarized deformation quantization

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Abstract

Let \mathcal{A} be a star product on a symplectic manifold (M, ω_0) , $\frac{1}{\hbar}[\omega]$ its Fedosov class, where ω is a deformation of ω_0 . We prove that for a complex polarization of ω there exists a commutative subalgebra, \mathcal{O} , in \mathcal{A} that is isomorphic to the algebra of functions constant along the polarization.

Let $F(\mathcal{A})$ consists of elements of \mathcal{A} whose commutator with \mathcal{O} belongs to \mathcal{O} . Then, $F(\mathcal{A})$ is a Lie algebra which is an \mathcal{O} -extension of the Lie algebra of derivations of \mathcal{O} . We prove a formula which relates the class of this extension, the Fedosov class, and the Chern class of P .

1 Introduction

Let (M, ω_0) be a symplectic manifold, C_M^∞ the sheaf of complex valued functions on M . A deformation quantization (or a star-product) on M is a structure of associative algebra on the sheaf of formal power series $C_M^\infty[[\hbar]]$ with the multiplication of the form

$$f * g = \sum_{i \geq 0} \hbar^i \mu_i(f, g), \quad f, g \in C_M^\infty, \quad (1)$$

where all μ_i are bidifferential operators, $\mu_0(f, g) = fg$ and $\mu_1(f, g) - \mu_1(g, f) = \{f, g\}$, the Poisson bracket inverse to ω_0 .

Two deformation quantizations \mathcal{A}_1 and \mathcal{A}_2 on (M, ω_0) are equivalent if there exists a power series $B = 1 + \hbar B_1 + \dots$, where B_i are differential operators, such that $B : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ is an isomorphism of algebras.

It is known that all equivalence classes of star-products on M with the Poisson bracket $\varphi_0 = \omega_0^{-1}$ can be obtained by the Fedosov method. According to this method, one constructs a flat connection, D , (called the Fedosov connection) on the bundle of Weyl algebras on M . The quantized algebra, \mathcal{A} , is realized as the subalgebra of flat sections of

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the Weyl algebra. The Weyl curvature of D , being a closed scalar two-form of the view $\omega = \omega_0 + t\omega_1 + \dots$ defines the Fedosov class

$$\theta(\mathcal{A}) = \frac{1}{t}[\omega] \in \frac{1}{t}[\omega_0] + H^2(M, \mathbb{C}[[t]]). \quad (2)$$

It is also known that the correspondence $\mathcal{A} \mapsto \theta(\mathcal{A})$ is a bijection between the set of isomorphism classes of star-products on (M, ω_0) and the set $\frac{1}{t}[\omega_0] + H^2(M, \mathbb{C}[[t]])$ modulo the group of formal symplectomorphisms of M , [Fe], [NT], [Xu].

Let P be a polarization of (M, ω_0) . That is, P is an integrable Lagrangean subbundle of the complexified tangent bundle $T_M^{\mathbb{C}}$. Let $\omega = \omega_0 + t\omega_1 + \dots$ be a closed form which is a deformation of ω_0 and \mathcal{P} a polarization of ω , which is a deformation of polarization P . We say that the triple (M, ω, \mathcal{P}) is a deformation of the triple (M, ω_0, P) .

Definition 1.1. A polarized star-product (or quantization) on (M, ω_0, P) is a pair $(\mathcal{A}, \mathcal{O})$ of $\mathbb{C}[[t]]$ algebras satisfying the conditions:

- 1) \mathcal{A} is a star-product on (M, ω_0) ,
- 2) there exists a deformation (M, ω, \mathcal{P}) of (M, ω_0, P) such that \mathcal{O} is the sheaf of functions constant along \mathcal{P} ,
- 3) the star-product in \mathcal{A} being restricted to \mathcal{O} coincides with the original commutative multiplication.

Two polarized star-products $(\mathcal{A}_1, \mathcal{O}_1)$ and $(\mathcal{A}_2, \mathcal{O}_2)$ are equivalent if there exists an isomorphism $B : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ of star-products such that $B(\mathcal{O}_1) = \mathcal{O}_2$.

The first result of the paper is the following theorem (cf. Theorem 2.10):

Theorem 1.2. For any triple (M, ω_0, P) , where P is good (see Definition 2.4) and any its deformation (M, ω, \mathcal{P}) there exists a polarized star-product, $(\mathcal{A}, \mathcal{O})$, such that \mathcal{O} is the sheaf of functions constant along \mathcal{P} and $\theta(\mathcal{A}) = \frac{1}{t}[\omega]$.

For example, any Kähler or real polarization is good. This result generalizes a result of N. Reshetikhin and M. Yakimov, [RY], who consider the case of a real polarization P defined by a Lagrangean fiber bundle $M \rightarrow B$.

Our proof of this result uses the Fedosov method adapted for the case with polarization. The analogous method was used by M. Bordemann and S. Waldmann, [BW], for a construction of quantization with separation of variables on a Kähler manifold.

In fact, we prove a slightly stronger statement. Namely, the polarized quantization $(\mathcal{A}, \mathcal{O})$ we constructed satisfies the property $f * g = fg$ for $f \in \mathcal{O}$ and any $g \in \mathcal{A}$. Such a type of star-products is in the spirit of A. Karabegov (see [Ka1], where he constructs star-products with separation of variables on Kähler manifolds).

Let $(\mathcal{A}, \mathcal{O})$ be a polarized quantization on (M, ω_0, P) . Let $F(\mathcal{A}) = \{f \in \mathcal{A}; [f, \mathcal{O}] \in \mathcal{O}\}$. The sheaf of \mathcal{O} modules $F(\mathcal{A})$ has the natural structure of a sheaf of Lie algebras with \mathcal{O} as a commutative ideal. $T_{\mathcal{O}} = F(\mathcal{A})/\mathcal{O}$ can be considered as a sheaf of Lie algebras consisting of derivations of \mathcal{O} , and we may consider the exact sequence of sheaves of \mathcal{O} modules and Lie algebras

$$0 \longrightarrow \mathcal{O} \longrightarrow F(\mathcal{A}) \longrightarrow T_{\mathcal{O}} \longrightarrow 0. \quad (3)$$

$F(\mathcal{A})$ is an example of an \mathcal{O} -extension of $T_{\mathcal{O}}$, (see Definition 4.1).. We prove that if P is a em strong polarization (see Definition 3.1), this sequence locally splits. In this case the class $\text{cl}(\mathcal{A}, \mathcal{O}) \in H^2(M, \frac{1}{t}\mathbb{C}[[t]])$ of this extension is defined.

The second result of the paper is a formula that relates three classes in $H^2(M, \frac{1}{t}\mathbb{C}[[t]])$ associated with a polarized deformation quantization $(\mathcal{A}, \mathcal{O})$: the Fedosov class $\theta(\mathcal{A})$, the Chern class of the polarization P , and $\text{cl}(\mathcal{A}, \mathcal{O})$.

Theorem 1.3. *Let $(\mathcal{A}, \mathcal{O})$ be a polarized quantization of (M, ω_0, P) . Then,*

$$\theta(\mathcal{A}) = \frac{1}{t}\text{cl}(\mathcal{A}, \mathcal{O}) - \frac{1}{2}c_1(P). \quad (4)$$

One can show that in the case of quantization on a Kähler manifold with separation of variables the class $\text{cl}(\mathcal{A}, \mathcal{O})$ coincides with the class defined by Karabegov in [Ka1]. The formula analogous to (4) relating the Karabegov and Fedosov classes was obtained in [Ka2].

The paper is organized as follows. In Section 2 we develop a variant of Fedosov's method adapted to the polarized setting and prove Theorem 1.2.

In Section 3 we define *strong* polarizations and study their local properties. We show that if P is a strong polarization, then any deformed polarization, \mathcal{P} , $\mathcal{P}_0 = P$, is also strong, and any strong polarization is good. We prove that any polarized quantization of (M, ω_0, P) with P a strong polarization is locally isomorphic to a standard quantization, the polarized Moyal star-product.

In Section 4 we prove Theorem 1.3. To this end we consider, for a given polarized quantization $(\mathcal{A}, \mathcal{O})$, the extension (3) and study its behavior under passing to the opposite quantization, $(\mathcal{A}^{op}, \mathcal{O})$, and to the quantization $(\mathcal{A}^\sigma, \mathcal{O})$ obtained by the change of variable $t \mapsto -t$. Furthermore, we compare classes $\text{cl}(\mathcal{A}, \mathcal{O})$ with the classes introduced by Deligne in [De].

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2 Polarized quantization

2.1 Setting and notation

Let M be a smooth manifold, C_M^∞ the sheaf of smooth complex valued functions on it. Let $\mathbb{C}[[t]]$ be the algebra of formal power series over \mathbb{C} . For a sheaf E of C_M^∞ modules, $E[[t]]$ denotes the sheaf $E \otimes_{\mathbb{C}} \mathbb{C}[[t]]$ completed in the t -adic topology. $E[[t]]$ is a sheaf of $C_M^\infty[[t]]$ modules.

Let \mathcal{E} be a locally free sheaf of $C_M^\infty[[t]]$ modules of finite rank. Denote by $T^k(\mathcal{E})$ the k -th tensor power of \mathcal{E} over $C_M^\infty[[t]]$ and by $T(\mathcal{E})$ the corresponding tensor algebra completed in the $\{\mathcal{E}, t\}$ -adic topology. Similarly we define the completed symmetric algebra $S(\mathcal{E})$. For a subsheaf \mathcal{P} of \mathcal{E} we denote by $\text{sym}_{\mathcal{P}} : S(\mathcal{P}) \rightarrow T(\mathcal{E})$ the natural map of $C_M^\infty[[t]]$ modules defined by symmetrization.

Let $\Lambda(\mathcal{E})$ be the exterior algebra of \mathcal{E} over $C_M^\infty[[t]]$. We will consider $T \otimes \Lambda = T(\mathcal{E}) \otimes_{C_M^\infty[[t]]} \Lambda(\mathcal{E})$ as a graded super-algebra regarding a section $x \in T(\mathcal{E}) \otimes \Lambda^k(\mathcal{E})$ of degree k even (odd) if k is even (odd).

Denote by $\delta_T = \delta_{T(\mathcal{E})}$ the continuous $C_M^\infty[[t]]$ linear derivation of $T \otimes \Lambda$ generated by the map $T^1(\mathcal{E}) \otimes 1 \rightarrow 1 \otimes \Lambda^1(\mathcal{E})$, $v \otimes 1 \mapsto 1 \otimes v$, $v \in \mathcal{E}$ is a section. It is clear that δ_T is a derivation of degree 1 and $\delta_T^2 = 0$. In the same way define a derivation $\delta_{\mathcal{E}} = \delta_{S(\mathcal{E})}$ in the (super-)algebra $S \otimes \Lambda = S(\mathcal{E}) \otimes \Lambda(\mathcal{E})$. It is easy to see that for a subsheaf $\mathcal{P} \subset \mathcal{E}$, δ_T on $T(\mathcal{E}) \otimes \Lambda(\mathcal{E})$ can be restricted to $S(\mathcal{P}) \otimes \Lambda(\mathcal{P})$ via the embedding $\text{sym}_{\mathcal{P}} \otimes \text{id}_{\mathcal{P}}$, and coincides with the $\delta_{S(\mathcal{P})}$ defined on the algebra $S(\mathcal{P}) \otimes \Lambda(\mathcal{P})$.

On the algebra $S \otimes \Lambda$ there is defined another derivation, δ^* , of degree -1 generated by the map $1 \otimes \Lambda^1 \rightarrow S^1 \otimes 1$, $1 \otimes v \rightarrow v \otimes 1$. It is easy to check that $(\delta^*)^2 = 0$ and $[\delta, \delta^*] = \delta\delta^* + \delta^*\delta = \text{deg}$, where deg is the derivation assigning to an element $x \in S^p \otimes \Lambda^q$ the element $(p+q)x$.

Let \mathcal{E} be presented as a direct sum of $C_M^\infty[[t]]$ submodules, $\mathcal{E} = \mathcal{P} \oplus \mathcal{Q}$. We will identify the tensor product of algebras, $(S(\mathcal{P}) \otimes \Lambda(\mathcal{P})) \otimes (S(\mathcal{Q}) \otimes \Lambda(\mathcal{Q}))$, with $(S(\mathcal{P}) \otimes S(\mathcal{Q})) \otimes (\Lambda(\mathcal{P}) \otimes \Lambda(\mathcal{Q})) = (S(\mathcal{P}) \otimes S(\mathcal{Q})) \otimes \Lambda(\mathcal{E})$, where the last identification is obtained via skew-symmetrization of the last factor. Denote $S(\mathcal{P}, \mathcal{Q}) = S(\mathcal{P}) \otimes S(\mathcal{Q})$, $\mathbf{S} = \mathbf{S}(\mathcal{P}, \mathcal{Q}) = (S(\mathcal{P}) \otimes S(\mathcal{Q})) \otimes \Lambda(\mathcal{E})$, $\mathbf{S}(\mathcal{P}) = S(\mathcal{P}) \otimes 1 \otimes \Lambda(\mathcal{E})$, and $\mathbf{S}(\mathcal{Q}) = 1 \otimes S(\mathcal{Q}) \otimes \Lambda(\mathcal{E})$.

One has the embedding

$$\text{sym}_{\mathcal{P}} \otimes \text{sym}_{\mathcal{Q}} \otimes \text{id} : (S(\mathcal{P}) \otimes S(\mathcal{Q})) \otimes \Lambda(\mathcal{E}) \rightarrow T(\mathcal{E}) \otimes \Lambda(\mathcal{E}). \quad (5)$$

It is obvious that the derivations $\delta_{\mathcal{P}}$, $\delta_{\mathcal{P}}^*$, $\delta_{\mathcal{Q}}$, $\delta_{\mathcal{Q}}^*$ induce the corresponding derivations on the algebra \mathbf{S} , and $\delta = \delta_{\mathcal{P}} + \delta_{\mathcal{Q}}$ coincides with the restriction to \mathbf{S} of δ_T on $T(\mathcal{E}) \otimes \Lambda(\mathcal{E})$ via the embedding (5).

Let us define the operator $\delta^{-1} = \delta_{\mathcal{P}, \mathcal{Q}}^{-1}$ on \mathbf{S} . There is the continuous $C_M^\infty[[t]]$ linear map that is equal to zero for $x \in C_M^\infty[[t]]$ and $\delta^{-1}(x) = (1/(p+r+q))\delta^*(x)$ for $x \in (S^p(\mathcal{P}) \otimes S^r(\mathcal{Q})) \otimes \Lambda^q(\mathcal{E})$, $p+r+q > 0$.

There is the obvious relation

$$\delta \delta^{-1} + \delta^{-1} \delta = \text{projection on } \mathbf{S}^+ \text{ along } C_M^\infty[[t]], \quad (6)$$

where \mathbf{S}^+ is the closure of $\oplus_{p+r+q>0} (S^p(\mathcal{P}) \otimes S^r(\mathcal{Q})) \otimes \Lambda^q(\mathcal{E})$.

2.2 Fedosov algebra

Let $\varphi : \mathcal{E} \otimes \mathcal{E} \rightarrow C_M^\infty[[t]]$ be a $C_M^\infty[[t]]$ linear skew-symmetric form and I the closed ideal in $T(\mathcal{E})$ generated by relations

$$x \otimes y - y \otimes x = t\varphi(x, y). \quad (7)$$

We call $\mathcal{W}(\mathcal{E}) = T(\mathcal{E})/I$ the Weyl algebra and $\mathbf{W} = \mathbf{W}(\mathcal{E}) = \mathcal{W} \otimes \Lambda(\mathcal{E})$ the Fedosov algebra over \mathcal{E} .

Let $\mathcal{E} = \mathcal{P} \oplus \mathcal{Q}$ be a decomposition into $C_M^\infty[[t]]$ modules.

The derivation δ of $T(\mathcal{E}) \otimes \Lambda(\mathcal{E})$ induces a derivation of \mathbf{W} . Indeed, δ applied to the both sides of (7) gives zero.

Define the Wick map, $\mathbf{w} = \mathbf{w}_{\mathcal{P}, \mathcal{Q}}$, as the composition $\mathbf{S}(\mathcal{P}, \mathcal{Q}) \rightarrow T(\mathcal{E}) \otimes \Lambda(\mathcal{E}) \rightarrow \mathbf{W}$, where the first map is (5) and the second is the projection. By the PBW theorem \mathbf{w} is an isomorphism of $C_M^\infty[[t]]$ modules.

Due to the isomorphism \mathbf{w} , all the operators $\delta_{\mathcal{P}}$, $\delta_{\mathcal{Q}}$, $\delta_{\mathcal{P}, \mathcal{Q}}^{-1}$ carry over from \mathbf{S} to \mathbf{W} . We retain for them the same notation. Note that while $\delta_{\mathcal{P}} + \delta_{\mathcal{Q}}$ does not depend on the decomposition and coincides with the δ induced from $T(\mathcal{E}) \otimes \Lambda(\mathcal{E})$, $\delta_{\mathcal{P}, \mathcal{Q}}^{-1}$ is not a derivation and does depend on the decomposition. In particular, one can suppose that the decomposition is trivial, $\mathcal{E} = \mathcal{E} \oplus 0$. In this case we denote $\delta_{\mathcal{E}}^{-1} = \delta_{\mathcal{E}, 0}^{-1}$.

Note that $\delta_{\mathcal{P}}$, $\delta_{\mathcal{Q}}$, and δ are derivations on \mathbf{W} .

Proposition 2.1. *One has*

$$H(\mathbf{W}, \delta) = C_M^\infty[[t]].$$

Moreover, if $x \in \mathcal{W}(\mathcal{E}) \otimes \Lambda^{k>0}(\mathcal{E})$ then $y = \delta_{\mathcal{P}, \mathcal{Q}}^{-1}x$ is such that $\delta y = x$ for any decomposition $\mathcal{E} = \mathcal{P} \oplus \mathcal{Q}$.

Proof. Follows from (6). □

2.3 Lie subalgebras in \mathcal{W}

Let $\mathcal{E} = \mathcal{P} \oplus \mathcal{Q}$ be a decomposition. We say that $x \in \mathbf{W}$ has $\mathbf{w}_{\mathcal{P}, \mathcal{Q}}$ -degree (p, q) if $\mathbf{w}_{\mathcal{P}, \mathcal{Q}}^{-1}(x) \in (S^p(\mathcal{P} \otimes S^q(\mathcal{Q})) \otimes \Lambda(\mathcal{E}))$. We say that $x \in \mathbf{W}$ has $\mathbf{w}_{\mathcal{P}, \mathcal{Q}}$ -degree k if $\mathbf{w}_{\mathcal{P}, \mathcal{Q}}^{-1}(x) \in \bigoplus_{p+q=k} (S^p(\mathcal{P}) \otimes S^q(\mathcal{Q})) \otimes \Lambda(\mathcal{E})$. The $\mathbf{w}_{\mathcal{E}}$ -degree is $\mathbf{w}_{\mathcal{E}, 0}$ -degree for the trivial decomposition $\mathcal{E} = \mathcal{E} \oplus 0$.

Let φ be nondegenerate. Let \mathfrak{g} be a sheaf of Lie algebras acting on \mathcal{E} . We call a $C_M^\infty[[t]]$ linear map $\lambda : \mathfrak{g} \rightarrow \mathcal{W}$ a realization of \mathfrak{g} if it is a Lie algebra morphism (\mathcal{W} is considered as a Lie algebra with respect to commutator $\frac{1}{t}[\cdot, \cdot]$) and for any $x \in \mathfrak{g}$ and $e \in \mathcal{E}$ one has $x(e) = \frac{1}{t}[\lambda(x), e]$. Any two realizations differ by a Lie algebra morphism of \mathfrak{g} to the center of \mathcal{W} , so if \mathfrak{g} is a sheaf of semisimple Lie algebras there is not more than one realization of \mathfrak{g} .

Denote by $\mathfrak{sp}(\mathcal{E})$ the sheaf of symplectic Lie algebras with respect to φ . Since $\mathfrak{sp}(\mathcal{E})$ is semisimple, there is a unique realization $\rho_{\mathcal{E}} : \mathfrak{sp}(\mathcal{E}) \rightarrow \mathcal{W}$. The image of this realization consists of elements having $\mathbf{w}_{\mathcal{E}}$ -degree two.

Let $\mathcal{E} = \mathcal{P} \oplus \mathcal{Q}$ be a decomposition into Lagrangean subsheaves. Denote by $\mathfrak{sp}(\mathcal{P}, \mathcal{E})$ the subsheaf of $\mathfrak{sp}(\mathcal{E})$ preserving \mathcal{P} . It is easy to check that $\mathfrak{sp}(\mathcal{P}, \mathcal{E})$ can be realized as the subset of elements of \mathcal{W} having $\mathbf{w}_{\mathcal{P}, \mathcal{Q}}$ -degree $(1, 1)$ and $(2, 0)$. Denote this realization by $\rho_{\mathcal{P}, \mathcal{E}} : \mathfrak{sp}(\mathcal{P}, \mathcal{E}) \rightarrow \mathcal{W}$. On the other hand, $\mathfrak{sp}(\mathcal{P}, \mathcal{E})$ can be realized in \mathcal{W} by $\rho_{\mathcal{E}}$.

Let us define a $\mathbb{C}[[t]]$ linear map $tr : \mathfrak{sp}(\mathcal{P}, \mathcal{E}) \rightarrow \mathbb{C}[[t]]$ in the following way. Let $a \in \mathfrak{sp}(\mathcal{P}, \mathcal{E})$ and a' its restriction to \mathcal{P} . We put $tr(a) = trace(a')$. It is easy to see that $\rho_{\mathcal{P}, \mathcal{E}}(a')$ (here a' is trivially extended to \mathcal{E}) is the $(1, 1)$ component of $\rho_{\mathcal{P}, \mathcal{E}}(a)$.

Lemma 2.2. *Let $\mathcal{E} = \mathcal{P} \oplus \mathcal{Q}$ be a decomposition into Lagrangean subsheaves. Let $a \in \mathfrak{sp}(\mathcal{P}, \mathcal{E})$. Then*

$$\rho_{\mathcal{P}, \mathcal{E}}(a) - t \frac{1}{2} tr(a) = \rho_{\mathcal{E}}(a).$$

Proof. Straightforward. □

2.4 Filtrations on \mathcal{W}

We define two decreasing filtrations on \mathcal{W} numbered by nonnegative integers.

The T -filtration $F_{\bullet}^T \mathcal{W}$ is defined as follows. We ascribe to the elements of \mathcal{E} degree 1 and to t degree 2. Then $F_n^T \mathcal{W}$ consists of elements of \mathcal{W} having the leading term of total degree $\geq n$.

The \mathcal{P} -filtration, $F_{\bullet}^{\mathcal{P}} \mathcal{W}$, is firstly defined on $S(\mathcal{P}) \otimes S(\mathcal{Q})$ by the subsets $F_n^{\mathcal{P}} = S^n(\mathcal{P})S(\mathcal{P}) \otimes S(\mathcal{Q})$, $n = 0, 1, \dots$, and carried over to \mathcal{W} via the Wick isomorphism.

We extend those filtrations to \mathbf{W} in the natural way standing, for example, $F_n^T \mathbf{W} = F_n^T \mathcal{W} \otimes \Lambda(\mathcal{E})$. We will use the following mnemonic notation. To point out, for example, that a section $x \in \mathbf{W}$ belongs to $F_n^T \mathbf{W}$ we write $F^T(x) \geq n$.

We call a subsheaf $\mathcal{P} \subset \mathcal{E}$ a φ -null subsheaf if φ when restricted to \mathcal{P} equals zero.

Proposition 2.3. *Let $\mathcal{E} = \mathcal{P} \oplus \mathcal{Q}$ be a decomposition of \mathcal{E} with \mathcal{P} a φ -null subsheaf. Then*

a) *The Wick map $\mathbf{w} : \mathbf{S} \rightarrow \mathbf{W}$ has the following property: for $a \in \mathbf{S}(\mathcal{P})$ and arbitrary $c \in \mathbf{S}$ one has $ac = \mathbf{w}(a)\mathbf{w}(c)$.*

The filtrations on \mathbf{W} have the properties:

b) *for $x, y \in \mathbf{W}$, if $F^{\mathcal{P}}(x) \geq k$ then $F^{\mathcal{P}}(xy) \geq k$;*

c) *$F^{\mathcal{P}}(\delta_{\mathcal{P}, \mathcal{Q}}^{-1}x) \geq F^{\mathcal{P}}(x)$;*

d) *$F^T(\delta_{\mathcal{P}, \mathcal{Q}}^{-1}x) > F^T(x)$.*

Proof. a) follows from the fact that \mathcal{P} is a Lagrangean subsheaf of \mathcal{E} and the definition of the Wick map. b) follows from a), c) and d) are obvious. \square

2.5 Good polarizations and connections

Let T_M be the complexified tangent bundle over M , $\mathcal{T} = T_M[[t]]$, and $\omega : \mathcal{T}^{\otimes 2} \rightarrow C_M^{\infty}[[t]]$ a nondegenerate symplectic form. It means that $\omega = \omega_0 + t\omega_1 + \dots$ where ω_0 is nondegenerate and $d\omega = 0$. Let $\varphi = \varphi_0 + t\varphi_1 + \dots$ be the nondegenerate Poisson bracket inverse to ω . It may be considered as the map $\varphi : (\mathcal{T}^*)^{\otimes 2} \rightarrow C_M^{\infty}[[t]]$, where $\mathcal{T}^* = T_M^*[[t]]$.

For any function $f \in C_M^{\infty}[[t]]$ denote by X_f the corresponding Hamiltonian vector field on M , $X_f = \varphi(df, \cdot)$. The map $\phi : \mathcal{T}^* \rightarrow \mathcal{T}$ defined by $df \mapsto X_f$ is an isomorphism of sheaves. Its inverse map is given by $X_f \mapsto \omega(\cdot, X_f) = df$.

Definition 2.4. a) *A direct subsheaf $\mathcal{P} \subset \mathcal{T}$ is called Lagrangean if $\omega(\mathcal{P}, \mathcal{P}) = 0$ and $\text{rank}(\mathcal{P}) = \frac{1}{2} \dim M = n$.*

b) *\mathcal{P} is called integrable if locally there exist functions a_1, \dots, a_n such that the Hamiltonian vector fields X_{a_1}, \dots, X_{a_n} form a local basis in \mathcal{P} .*

c) *A Lagrangean integrable \mathcal{P} is called a polarization of ω . In this case all the X_{a_i} , $i = 1, \dots, n$, pairwise commute, and da_i form a local basis in \mathcal{P}^{\perp} .*

d) *A polarization \mathcal{P} is called good if there exist additional functions f_1, \dots, f_n such that the Hamiltonian vector fields X_{a_i}, X_{f_i} , $i = 1, \dots, n$, pairwise commute and form a local basis in \mathcal{T} .*

Definition 2.5. Let $\mathcal{P} \subset \mathcal{T}$ be a Lagrangean subsheaf. We call a connection, ∇ , on M a \mathcal{P} -connection if

- a) it preserves ω and is torsion free, i.e. is a symplectic connection, and
- b) it preserves \mathcal{P} , i.e. $\nabla(\mathcal{P}) \subset \mathcal{P} \otimes \Lambda^2(\mathcal{T}^*)$, and is flat on \mathcal{P} along \mathcal{P} , i.e. for any $X, Y \in \mathcal{P}$ one has $(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]})(\mathcal{P}) = 0$.

Proposition 2.6. Let $\mathcal{P} \subset \mathcal{T}$ be a good polarization of ω . Then, there exists a \mathcal{P} -connection on M .

Proof. Let a_1, \dots, a_{2n} be functions such that $X_i = X_{a_i}$, $i = 1, \dots, n$, form a local basis in \mathcal{P} and all the X_i commute and form a local basis in \mathcal{T} . Let us set $\nabla_{X_i} X_j = [X_i, X_j] = 0$. This rule defines a local connection.

Let us prove the invariance of φ , i.e.

$$\nabla_z(\omega(x, y)) = \omega(\nabla_z x, y) + \omega(x, \nabla_z y) \text{ for any } x, y, z \in \mathcal{T}.$$

But for $x = X_i$, $y = X_j$, $z = X_k$ this equation is equivalent to the relation

$$\varphi(a_k, \varphi(a_i, a_j)) = \varphi(\varphi(a_k, a_i), a_j) + \varphi(a_i, \varphi(a_k, a_j)) = 0,$$

which holds by the Jacobi identity for the Poisson bracket φ .

Let us proof that ∇ is torsion free. Since the torsion is $C_M^\infty[[t]]$ linear, it is enough to prove that

$$\nabla_{X_i}(X_j) - \nabla_{X_j}(X_i) - [X_i, X_j] = 0$$

for all pairs X_i, X_j of our basis. But this follows from the definition of ∇ and from the fact that all X_i pairwise commute. Similarly, from the fact that all X_i pairwise commute follows that ∇ is flat.

That ∇ preserves \mathcal{P} is obvious.

Since $X_f \in \mathcal{P}$ is equivalent to $df \in \mathcal{P}^\perp$, it is obvious that ∇ has the property: for any $f, g \in C_M^\infty[[t]]$ such that $X_f, X_g \in \mathcal{P}$ one has

$$\nabla_{X_f} X_g = 0. \tag{8}$$

Now, let us prove the existence of a global connection.

Let $\{U_\alpha\}$ is an open covering of M such that on each U_α there is a \mathcal{P} -connection ∇_α . Then the differences $\nabla_\alpha - \nabla_\beta$ form on $U_\alpha \cap U_\beta$ a Čech cocycle $\psi_{\alpha, \beta} \in \text{Hom}(\mathcal{T} \otimes \mathcal{T}, \mathcal{T})$, $\psi_{\alpha, \beta}(X, Y) = \nabla_\alpha X Y - \nabla_\beta X Y$.

$\psi_{\alpha, \beta}$ satisfy the following properties.

It follows from (8) that

$$\psi_{\alpha, \beta}(X, Y) = 0 \quad \text{for } X, Y \in \mathcal{P}. \tag{9}$$

Since all ∇_α are torsion free, $\psi_{\alpha, \beta}$ are symmetric. Since all ∇_α preserve \mathcal{P} , one has $\psi_{\alpha, \beta}(X, Y) \in \mathcal{P}$ for $Y \in \mathcal{P}$.

In addition, $\psi_{\alpha, \beta}$ considered as elements from $\text{Hom}(\mathcal{T}, \text{Hom}(\mathcal{T}, \mathcal{T}))$, $X \mapsto \psi_{\alpha, \beta}(X, \cdot)$, belong to $\text{Hom}(\mathcal{T}, \text{sp}(\mathcal{T}))$ where $\text{sp}(\mathcal{T})$ consists of endomorphisms of \mathcal{T} preserving ω invariant.

Since all the properties above are $C_M^\infty[[t]]$ linear, it is possible to find tensors $\psi_\alpha \in \text{Hom}(\mathcal{T} \otimes \mathcal{T}, \mathcal{T})$ satisfying the same properties and such that $\psi_\alpha - \psi_\beta = \psi_{\alpha,\beta}$. Then $\nabla = \nabla_\alpha - \psi_\alpha = \nabla_\beta - \psi_\beta$ is a globally defined connection. Flatness of ∇ on \mathcal{P} along \mathcal{P} follows from the fact that for all α $\psi_\alpha(X, Y) = 0$ for $X, Y \in \mathcal{P}$, which follows from (9). ∇ is torsion free because all ψ_α are symmetric. So, ∇ satisfies the proposition. \square

2.6 Fedosov's construction

Let (M, ω_0) be a symplectic manifold. It is known that all equivalent classes of star-products on M with the Poisson bracket $\varphi_0 = \omega_0^{-1}$ can be obtained by the Fedosov method. According to this method, one constructs a flat connection, D , (called the Fedosov connection) in the Weyl algebra defined on the cotangent bundle with the help of relations (7), where $\varphi = \varphi_0$. The quantized algebra, \mathcal{A} , is realized as the subalgebra of flat sections of the Weyl algebra. The Weyl curvature of D , being a closed scalar two-form of the view $\tilde{\theta} = \omega_0 + t\omega_1 + \dots$ defines the Fedosov class

$$\theta(\mathcal{A}) = \frac{1}{t}[\tilde{\theta}] \in \frac{1}{t}[\omega_0] + H^2(M, \mathbb{C}[[t]]). \quad (10)$$

It is also known that the correspondence $\mathcal{A} \mapsto [\theta(\mathcal{A})]$ is a bijection between the set of isomorphism classes of star-products on (M, ω_0) and the set $\frac{1}{t}[\omega_0] + H^2(M, \mathbb{C}[[t]])$ modulo the group of formal symplectomorphisms of M , [Fe], [NT], [Xu].

We will adapt the Fedosov method to construct a polarized star-product corresponding to a class $\frac{1}{t}[\tilde{\theta}] \in \frac{1}{t}[\omega_0] + H^2(M, \mathbb{C}[[t]])$ with a polarization of $\tilde{\theta}$. We will see that in the presence of polarization by realizing the Fedosov scheme there appears also a Wick curvature which differs from the Weyl curvature by t times a half of the first Chern class of the polarization.

Let $\mathcal{P}' \subset \mathcal{T}$ be a good polarization of ω . It is easy to prove that there exists a complement Lagrangean subsheaf $\mathcal{Q}' \subset \mathcal{T}$ such that $\mathcal{T} = \mathcal{P}' \oplus \mathcal{Q}'$ on M .

In the following we set $\mathcal{E} = \mathcal{T}^*$ and consider the decomposition $\mathcal{E} = \mathcal{P} \oplus \mathcal{Q}$, there \mathcal{P} and \mathcal{Q} correspond to \mathcal{P}' and \mathcal{Q}' by the isomorphism ϕ between \mathcal{T} and \mathcal{E} . Note that \mathcal{P} and \mathcal{P}' are orthogonal to each other, so that we may write $\mathcal{P}' = \mathcal{P}^\perp$.

Let ∇ be a \mathcal{P}^\perp -connection on M . Then the induced connection $\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes \Lambda^1 \mathcal{E}$ on \mathcal{E} preserves \mathcal{P} , i.e. $\nabla(\mathcal{P}) \subset \mathcal{P} \otimes \Lambda^2(\mathcal{E})$, and is flat on \mathcal{P} along \mathcal{P}' , i.e. for any $X, Y \in \mathcal{P}'$ one has $(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]})(\mathcal{P}) = 0$.

∇ gives a derivation of the Fedosov algebra $\mathbf{W} = \mathbf{W}(\mathcal{E})$, which is an extension of the de Rham differential on functions. Analogously, ∇ gives such derivations of the algebras $T(\mathcal{E}) \otimes \Lambda(\mathcal{E})$, $S(\mathcal{E}) \otimes \Lambda(\mathcal{E})$, and $(S(\mathcal{P}) \otimes S(\mathcal{Q})) \otimes \Lambda(\mathcal{E})$. These derivations commutes with the maps (5) and \mathbf{w} .

For convenience, we will mark the elements of the Fedosov algebra lying in $\mathcal{E} \otimes 1$ by letters with hat over them (\hat{x}), and by dx we will denote the copy of \hat{x} lying in $1 \otimes \Lambda^1 \mathcal{E}$.

It is easy to check that for $\tilde{\delta} = \omega_{ij} \hat{x}^i \otimes dx^j$ one has

$$\begin{aligned} \delta &= \frac{1}{t} \text{ad}(\tilde{\delta}), \\ \tilde{\delta}^2 &= t\omega. \end{aligned} \quad (11)$$

From the fact that the torsion of ∇ is equal to zero follows

$$\nabla(\tilde{\delta}) = 0. \quad (12)$$

Since ∇^2 is a $C_M^\infty[[t]]$ linear derivation of degree 0 preserving \mathcal{P} , there is an element $R \in \rho_{\mathcal{P}, \mathcal{E}}(\mathfrak{sp}) \otimes \Lambda^2(\mathcal{E})$ such that $\nabla^2 = \frac{1}{t}\text{ad}(R)$. In particular, one has to be

$$F^{\mathcal{P}}(R) \geq 1. \quad (13)$$

According to Fedosov, [Fe], we also define $R^F \in \rho_{\mathcal{E}}(\mathfrak{sp}) \otimes \Lambda^2(\mathcal{E})$ satisfying $\nabla^2 = \frac{1}{t}\text{ad}(R^F)$.

From (12) follows

$$\delta(R) = \delta(R^F) = 0. \quad (14)$$

Following to Fedosov, we will consider connections on \mathbf{W} of the form

$$D = \nabla + \frac{1}{t}\text{ad}(\gamma), \quad \gamma \in \mathcal{W} \otimes \Lambda^1(\mathcal{E}). \quad (15)$$

We define the Wick curvature of D as

$$\Omega_D = R + \nabla(\gamma) + \frac{1}{t}\gamma^2. \quad (16)$$

According to Fedosov, we also define the Weyl (or Fedosov) curvature of D as

$$\Omega_D^F = R^F + \nabla(\gamma) + \frac{1}{t}\gamma^2 = R - t\frac{1}{2}\text{tr}(R) + \nabla(\gamma) + \frac{1}{t}\gamma^2 = \Omega_D - t\frac{1}{2}\text{tr}(R), \quad (17)$$

where the second equality is due to Lemma 2.2.

Proposition 2.7. *$\text{tr}(R)$ is a closed form polarized by \mathcal{P} . The class of $\text{tr}(R)$ in $H^2(M, \mathbb{C})$ coincides with the first Chern class of \mathcal{P} , i.e. one has*

$$[\text{tr}(R)] = c_1(\mathcal{P}) = c_1(\mathcal{P}_{t=0}). \quad (18)$$

Proof. Since $\text{tr}(R)$ is a $C_M^\infty[[t]]$ valued 2-form, it follows from the Bianchi identity that it is a closed 2-form. From flatness of \mathcal{P} along \mathcal{P}^\perp follows that the $(1, 1)$ component of R does not contain terms of the view $R_{i,j} \otimes dy^i \wedge dy^j$ with $dy^i, dy^j \in \mathcal{Q}$. It follows that \mathcal{P} is a polarization of $\text{tr}(R)$.

It is easy to see that ∇ being restricted to \mathcal{P} defines a connection on \mathcal{P} , $\nabla_{\mathcal{P}}$, and

$$\nabla_{\mathcal{P}}^2 = \frac{1}{t}\text{ad}R_{1,1},$$

where $R_{1,1}$ is the $\mathbf{w}_{\mathcal{P}, \mathcal{Q}}$ -degree $(1, 1)$ component of the $\mathfrak{sp}_{\mathcal{P}, \mathcal{Q}}$ valued 2-form R . Now (18) follows from the definition of the Chern classes. Since Chern classes are integer valued elements of $H^2(M, \mathbb{C})$, i.e. belong to $H^2(M, \mathbb{Z})$, the independence of $c_1(\mathcal{P})$ on the parameter t is obvious. \square

One has

$$D^2 = \frac{1}{t} \text{ad}(\Omega_D) = \frac{1}{t} \text{ad}(\Omega_D^F). \quad (19)$$

Let us take γ in the form

$$\gamma = \tilde{\delta} + r, \quad r \in \mathcal{W} \otimes \Lambda^1(\mathcal{E}), \quad F^T(r) \geq 3. \quad (20)$$

Then the connection D has the form

$$\nabla + \delta + \frac{1}{t} \text{ad}(r). \quad (21)$$

Using (11) and (12), we obtain that its Wick curvature is

$$\Omega_D = R + \nabla(\tilde{\delta} + r) + \frac{1}{t}(\tilde{\delta} + r)^2 = \omega + \delta r + R + \nabla r + \frac{1}{t}r^2. \quad (22)$$

Proposition 2.8. *There exists an element $r \in \mathcal{W}(\mathcal{E}) \otimes \Lambda^1(\mathcal{E})$ such that*

- a) $F^T(r) \geq 3$;
- b) $F^{\mathcal{P}}(r) \geq 1$;
- c) *the connection $D = \nabla + \delta + \frac{1}{t} \text{ad}(r)$ is flat, i.e. $D^2 = 0$;*
- d) *for its Wick curvature one has*

$$\Omega_D = \omega;$$

- e) *for the class in $H^2(M, \mathbb{C}[[t]])$ of its Weyl curvature one has*

$$[\Omega_D^F] = [\Omega_D] - t \frac{1}{2} c_1(\mathcal{P}). \quad (23)$$

Proof. First of all, we apply the Fedosov method, ([Fe], Theorem 5.2.2), to find r satisfying d). According to (22), r must obey the equation

$$\delta r = -(R + \nabla r + \frac{1}{t}r^2) \quad (24)$$

Look for r as the limit of the sequence, $r = \lim r_k$, where $r_k \in \mathcal{W}(\mathcal{E}) \otimes \Lambda^1(\mathcal{E})$, $k = 3, 4, \dots$, and $F^T(r_k - r_{k-1}) \geq k$. As in Lemma 5.2.3 of [Fe], using Proposition (2.3) d) and the fact that $F^T(R) \geq 2$, such r_k can be calculated recursively:

$$\begin{aligned} r_3 &= -\delta_{\mathcal{P}, \mathcal{Q}}^{-1}(R) \\ r_{k+3} &= -(r_3 + \delta_{\mathcal{P}, \mathcal{Q}}^{-1}(\nabla r_{k+2} + \frac{1}{t}r_{k+2}^2)). \end{aligned} \quad (25)$$

So, a) and d) are proven. e) follows from (17) and Proposition 2.7.

Let us prove that $F^{\mathcal{P}}(r_k) \geq 1$ for all k . That $F^{\mathcal{P}}(r_3) \geq 1$ follows from the fact that $F^{\mathcal{P}}(R) \geq 1$ and from Proposition (2.3) c). Suppose that $F^{\mathcal{P}}(r_i) \geq 1$ for $i < k + 3$. Then $F^{\mathcal{P}}(\nabla r_{k+2}) \geq 1$ because ∇ preserves \mathcal{P} . On the other hand, $F^{\mathcal{P}}(r_{k+2}^2) \geq 1$ because of Proposition (2.3) b), therefore from (25) follows that $F^{\mathcal{P}}(r_{k+3}) \geq 1$ as well. So, we have that r being the limit of the convergent sequence r_k satisfy the conditions a), b), and d) of the proposition. c) obviously follows from d) and (19). \square

Denote by \mathcal{W}_D the subsheaf of \mathcal{W} consisting of flat sections a , i.e. such that $Da = 0$. Since D is a derivation of \mathbf{W} , it is clear that \mathcal{W}_D is a sheaf of subalgebras. Let $\sigma = id - (\delta\delta_{\mathcal{P},\mathcal{Q}}^{-1} + \delta_{\mathcal{P},\mathcal{Q}}^{-1}\delta)$. Then, as follows from (6), $\sigma : \mathcal{W} \rightarrow C_M^\infty[[t]]$ is a projection, where $C_M^\infty[[t]]$ considers as the center of the algebra \mathcal{W} .

Proposition 2.9. *a) The map $\sigma : \mathcal{W}_D \rightarrow C_M^\infty[[t]]$ is a bijection.*

b) The inverse map $\tau : C_M^\infty[[t]] \rightarrow \mathcal{W}_D$ has the form $\tau(f) = f + \hat{f}$, there $F^T(\hat{f}) > F^T(f)$.

c) If $df \in \mathcal{P}$ then $F^{\mathcal{P}}(\hat{f}) \geq 1$.

d) If $df \in \mathcal{P}$ then $\sigma(\tau(f)\tau(g)) = fg$ for any $g \in C_M^\infty[[t]]$.

Proof. Again, we apply the Fedosov iteration procedure. According to [Fe], Theorem 5.2.4, we look for $\tau(f)$ as a limit, $\tau(f) = \lim a_k$, there $a_k \in \mathcal{W}$ can be calculated recursively:

$$\begin{aligned} a_0 &= f \\ a_{k+1} &= a_0 + \delta_{\mathcal{P},\mathcal{Q}}^{-1}(\nabla a_k + \frac{1}{t}\text{adr}(a_k)). \end{aligned} \quad (26)$$

Put $\hat{f} = \tau(f) - a_0$. As in [Fe], Theorem 5.2.4, one proves that such $\tau(f)$ and \hat{f} satisfy a) and b). Now observe that $a_1 - a_0 = \delta_{\mathcal{P},\mathcal{Q}}^{-1}(1 \otimes df)$ and if $df \in \mathcal{P}$ then $F^{\mathcal{P}}(a_1 - a_0) \geq 1$. By induction we conclude that $F^{\mathcal{P}}(a_k - a_0) \geq 1$ for all $k \geq 1$. So $F^{\mathcal{P}}(a - a_0) \geq 1$ as well, which proves c).

Let us prove d). We have $\tau(f)\tau(g) = f\tau(g) + \hat{f}\tau(g)$. Since by c) $F^{\mathcal{P}}(\hat{f}) \geq 1$, $F^{\mathcal{P}}(\hat{f}\tau(g)) \geq 1$ as well. It follows that $\sigma(\hat{f}\tau(g)) = 0$ and $\sigma(\tau(f)\tau(g)) = \sigma(f\tau(g)) = fg$, because σ is a $C_M^\infty[[t]]$ linear map and $\sigma(\tau(g)) = g$. \square

2.7 Polarized star-product

Let T_M be the complexified tangent bundle over M , $\mathcal{T} = T_M[[t]]$, and $\omega : \mathcal{T}^{\otimes 2} \rightarrow C_M^\infty[[t]]$ a nondegenerate symplectic form, $\omega = \omega_0 + t\omega_1 + \dots$. Let $\mathcal{P} \subset \mathcal{T}$ be a polarization of ω . We say that ω is a deformation of the symplectic form ω_0 and \mathcal{P} is a deformation (or extension) of $\mathcal{P}_0 = \mathcal{P}/t\mathcal{P}$. It is clear that \mathcal{P}_0 is a polarization of ω_0 .

Let $\tilde{\omega}$ be another deformation of ω_0 . We say that ω and $\tilde{\omega}$ are equivalent if they define the same class in $\omega_0 + H^2(M, t\mathbb{C}[[t]])$.

Let \mathcal{P} be a polarization of ω which is an extension of \mathcal{P}_0 . It is easy to show that if $\tilde{\omega} = \omega + d\lambda$ and X_λ is a vector field corresponding to λ then $\tilde{\mathcal{P}} = \exp(L_{X_\lambda})\mathcal{P}$ is a polarization of $\tilde{\omega}$ extending \mathcal{P}_0 , which we call an equivalent polarization. Thus, it is meaningful to speak about a polarization of a class of $\omega_0 + H^2(M, t\mathbb{C}[[t]])$. It is the equivalence class of extensions of \mathcal{P}_0 .

As we have mentioned at the beginning of Section 2.6, the classification theorem for star-products say that all star-products on $C_M^\infty[[t]]$ with the Poisson bracket inverse to ω_0 are parameterized by elements

$$\theta = \frac{1}{t}[\tilde{\theta}] \in \frac{1}{t}[\omega_0] + H^2(M, \mathbb{C}[[t]]) \quad (27)$$

where $\tilde{\theta}$ is the Weyl scalar curvatures of the Fedosov connections.

Theorem 2.10. *Let (M, ω_0) be a symplectic manifold. Let ω be a deformation of ω_0 and \mathcal{P} a good polarization of ω . Let \mathcal{O} be the sheaf of functions from $C_M^\infty[[t]]$ constant along \mathcal{P} .*

Then, there exists a star-product, $$, on $C_M^\infty[[t]]$ having the Fedosov class $\frac{1}{t}[\omega]$ and satisfying the property:*

$$f * g = fg \quad \text{for all } f \in \mathcal{O}. \quad (28)$$

Proof. Let α be a 2-form representing $\frac{1}{2}c_1(\mathcal{P})$ and having \mathcal{P} as its polarization (see Proposition (2.7)). Construct the star-product for $\omega' = \omega + t\alpha$ and \mathcal{P} as in Proposition 2.9. According to Proposition 2.8 d), e), the Wick curvature corresponding to that star-product is equal to ω' and its Fedosov class is equal to $\frac{1}{t}[\omega]$.

Property (28) follows from Proposition 2.9 d). \square

3 Strong polarizations

3.1 Definition and deformation of a strong polarization

Let (M, ω_0) be a symplectic manifold of dimension $2n$, T the tangent bundle on M and $T \otimes_{\mathbb{R}} \mathbb{C} = T^{\mathbb{C}}$ its complexification. Let $P \subset T^{\mathbb{C}}$ be a polarization of ω_0 , i.e. P is a Lagrangean subbundle and locally there exist functions a_1, \dots, a_n such that da_1, \dots, da_n give a local basis for $P^\perp \subset (T^{\mathbb{C}})^*$. Let the triple (M, ω, \mathcal{P}) be a deformation of the triple (M, ω_0, P) , i.e. \mathcal{P} is a polarization of symplectic form $\omega = \omega_0 + t\omega_1 + \dots$, $\mathcal{P}_0 = P$. We are going to give simple conditions on P and \mathcal{P} which guarantee for them to be good polarizations.

Denote by $\mathcal{O} = \mathcal{O}_P \subset C_M^\infty$ the subsheaf of functions constant along the polarization P . In the obvious way one can form an analog of the Dolbeault complex, which we also call Dolbeault: $(\bigwedge^\bullet P^*, \bar{d})$, where \bar{d} is the differential along P . By definition, \mathcal{O} consists of functions f such that $\bar{d}f = 0$. Note that the Dolbeault complex is meaningful for any involutive P , i.e. when $[P, P] \subset P$, and for any involutive deformation \mathcal{P} .

Definition 3.1. *We call a polarization P strong if for any point $m \in M$ there exists a neighborhood U of m such that the complex of sections $(\Gamma(U; \bigwedge^\bullet P^*), \bar{d})$ is exact and gives a resolution of $\Gamma(U; \mathcal{O}_P)$.*

We give the same definition for a deformed polarization \mathcal{P} of ω .

Proposition 3.2. *Let P be a strong polarization of ω_0 . Let ω be a symplectic deformation of ω_0 and $\mathcal{P} \subset \mathcal{T} = T^{\mathbb{C}}[[t]]$ a deformation of P , $\mathcal{P}_0 = P$, which is Lagrangean with respect to ω and involutive, i.e. $[\mathcal{P}, \mathcal{P}] \subset \mathcal{P}$. Then \mathcal{P} is a strong polarization of ω .*

Proof. First of all, let us prove that \mathcal{P} is a polarization, i.e. \mathcal{P} is integrable. Since \mathcal{P} is involutive, the Dolbeault complex $(\bigwedge^\bullet \mathcal{P}^*, \bar{d}_t)$ is meaningful, where $\bar{d}_t = \bar{d} + t\bar{d}'$ is a deformation of the differential \bar{d} from the Dolbeault complex for P . Let U be a neighborhood such that the complex $(\Gamma_U(\bigwedge^\bullet P^*), \bar{d})$ is exact. It is enough to prove that if f is a function on U satisfying $\bar{d}f = 0$ then there is an extension $f_t = f + tf' \in C_U^\infty[[t]]$ such that $\bar{d}_t f_t = 0$.

Suppose that we have already found a series $f^{(n-1)} = f + tf_1 + \dots + t^{n-1}f_{n-1}$ such that $\bar{d}_t f^{(n-1)} = 0 \bmod t^n$. Then $\bar{d}_t f^{(n-1)} = t^n \gamma_n \bmod t^{n+1}$. Applying \bar{d}_t to the both sides of this equation and dividing by t^n we obtain $\bar{d}_t \gamma_n = 0 \bmod t$. Hence, $\bar{d}_t \gamma_n = 0$. So, γ_n is a closed 1-form in the Dolbeault complex corresponding to P , and one can find a function f_n such that $\bar{d}_t f_n = -\gamma_n$. It is clear that if we put $f^{(n)} = f^{(n-1)} + t^n f_n$, we obtain $\bar{d}_t f^{(n)} = 0 \bmod t^{n+1}$. In this way we construct step-by-step the series f_t such that $\bar{d}_t f_t = 0$ and prove that \mathcal{P} is a polarization of ω .

That \mathcal{P} is strong is obvious because of the upper-semicontinuity argument. \square

Proposition 3.2 shows that the deformed polarization \mathcal{P} is strong if $P = \mathcal{P}_0$ is strong. There is the following proposition provides sufficient conditions for a Lagrangean subbundle P to be a strong polarization.

Proposition 3.3. *Let (M, ω_0) be symplectic manifold with a Lagrangean subbundle $P \subset T_M^{\mathbb{C}}$. Suppose P satisfies the following conditions:*

- i) the subsheaf $P \cap \overline{P}$ is a subbundle in $T_M^{\mathbb{C}}$;*
- ii) $P + \overline{P}$ is involutive.*

Then P is a strong polarization of ω .

Proof. The integrability of P follows from the Frobenius-Nirenberg theorem. The exactness of the Dolbeault complex for P is a theorem due to Rawnsley ([Ra], Thm. 2). \square

Note that in the analytic case, i.e. when M is an analytic manifold and ω_0 is an analytic form, a Lagrangean involutive analytic subbundle P is strong. This follows from the Frobenius theorem.

3.2 Local properties of strong polarizations

From now on we will denote by P either a polarization of ω_0 or a deformed polarization of ω , and $\mathcal{O} = \mathcal{O}_P$ is the sheaf of functions constant along P . By $\{\cdot, \cdot\}$ we will denote the Poisson bracket corresponding to ω_0 or ω .

If P is integrable, there exist, locally, functions a_i , $i = 1, \dots, n$, such that $\{a_i, a_j\} = 0$ and $X_{a_i} = \{a_i, \cdot\}$ form a local basis in P . Denote by \bar{d}^i the dual basis in P^* . It is clear that \bar{d}^i is a closed form in the Dolbeault complex and

$$\bar{d} = \sum_i \{a_i, \cdot\} \bar{d}^i. \quad (29)$$

Lemma 3.4. *Suppose P is a strong polarization. Then, locally, there exist functions f_i such that $\{f_i, a_j\} = \delta_{ij}$.*

Proof. For any i , consider the closed form \bar{d}^i . By Definition 3.1, there is a function f_i such that $\bar{d}(f_i) = -\bar{d}^i$. By (29), f_i satisfies the lemma. \square

Let $F_{\mathcal{O}} = \{b \in C_M^{\infty}; \{b, \mathcal{O}\} \subset \mathcal{O}\}$. It follows from the Jacobi identity that $\{F_{\mathcal{O}}, F_{\mathcal{O}}\} \subset F_{\mathcal{O}}$. Hence, $F_{\mathcal{O}}$ is a Lie algebra acting on \mathcal{O} by derivations. The kernel of this action is equal to \mathcal{O} itself.

Note that $F_{\mathcal{O}}$ is a locally free \mathcal{O} module. Locally, it is freely generated as an \mathcal{O} module by functions $1, f_1, \dots, f_n$, where f_i are as in Lemma 3.4.

Let us denote by $\Omega^1 = \Omega_{\mathcal{O}}^1$ the \mathcal{O} -submodule of $(T^{\mathbb{C}})^*$ generated by the sheaf $d\mathcal{O}$, i.e. $\Omega^1 = \mathcal{O}d\mathcal{O}$. Let $T_{\mathcal{O}} = F_{\mathcal{O}}/\mathcal{O}$. It is a locally free \mathcal{O} module and is a sheaf of Lie algebras acting on \mathcal{O} with the local basis $\{f_i, \cdot\}$. It is easy to see that Ω^1 can be naturally identified with $\text{Hom}_{\mathcal{O}}(T_{\mathcal{O}}, \mathcal{O})$ and $d^i = da_i$ form the dual basis to $\{f_i, \cdot\}$.

We form in the obvious way an analog of de Rham complex: $(\bigwedge_{\mathcal{O}_P}^{\bullet} \Omega^1, d)$, which we call \mathcal{O}_P -de Rham complex.

Lemma 3.5. *Let P be a strong polarization. Then for any point of M there exists a neighborhood U such that the complex $(\Gamma_U(\bigwedge_{\mathcal{O}_P}^{\bullet} \Omega^1), d)$ is exact and gives a resolution for \mathbb{C} .*

Proof. Let us choose U such that the Dolbeault complex and the usual C^{∞} -de Rham complex $(\bigwedge^{\bullet}(T^{\mathbb{C}})^*, d_{dR})$ are exact. On the de Rham complex $(\bigwedge^{\bullet}(T^{\mathbb{C}})^*, d_{dR})$ consider the Hodge filtration generated by $P^{\perp} \subset (T^{\mathbb{C}})^*$. Considering the “stupid” filtration on $(\bigwedge_{\mathcal{O}_P}^{\bullet} \Omega^1, d)$ we have that the natural inclusion $(\bigwedge_{\mathcal{O}_P}^{\bullet} \Omega^1, d) \rightarrow (\bigwedge^{\bullet}(T^{\mathbb{C}})^*, d_{dR})$ is a filtered quasiisomorphism, i.e. induces a quasiisomorphism on the associated graded complexes. This follows from the fact that, since P is strong, the Dolbeault complex is exact. Now our lemma follows from the exactness of $(\bigwedge^{\bullet}(T^{\mathbb{C}})^*, d_{dR})$. \square

In the following, saying that the Dolbeault or \mathcal{O} -de Rham complex of sheaves is exact we mean that for any point of M there exists a neighborhood U such that the corresponding complex of sections over U is exact.

Lemma 3.6. *Let P be a strong polarization. Then f_i in Lemma 3.4 can be chosen in such a way that $\{f_i, f_j\} = 0$.*

Proof. By Lemma 3.4 one can choose g_i such that $\{g_i, a_j\} = \delta_{ij}$. For any functions b_1, \dots, b_n from \mathcal{O} the functions $f_i = g_i + b_i$ also satisfy Lemma 3.4. Let us find b_i such that $\{f_i, f_j\} = 0$. We find b_i from the equation

$$\{g_i + b_i, g_j + b_j\} = \{g_i, g_j\} + \{g_i, b_j\} - \{g_j, b_i\} = 0.$$

Note that $g = \sum_{i,j} \{g_i, g_j\} d^i \wedge d^j$ is a closed 2-form of the \mathcal{O}_P -de Rham complex and the equation can be rewritten as

$$db = -g,$$

where $b = \sum_i b_i d^i$. By Lemma 3.5, such b exists. \square

Lemma 3.6, in particular, means that locally there exists a polarization, P' , of ω complement to P . This polarization is defined as annihilator of df_1, \dots, df_n . It is obvious that locally the form ω can be written as $\omega = \sum_i df_i \wedge da_i$. One can prove that P' is also a strong polarization.

Proposition 3.7. *Let P be a strong polarization. Then*

- a) P is a good polarization and
- b) the sequence

$$0 \longrightarrow \mathcal{O}_P \longrightarrow F_{\mathcal{O}} \xrightarrow{\pi} T_{\mathcal{O}} \longrightarrow 0 \tag{30}$$

locally splits as an exact sequence of \mathcal{O} modules and Lie algebras.

Proof. a) follows from the existence of functions f_i satisfying Lemma 3.6.

b) One needs to construct, locally, a Lie algebra morphism $s : T_{\mathcal{O}} \rightarrow F_{\mathcal{O}}$ such that $\pi s = id$. Let us choose functions a_i and f_i , $i = 1, \dots, n$, as in Lemma 3.6 and assign $\{f_i, \cdot\} \mapsto f_i$. Since the elements $X_{f_i} = \{f_i, \cdot\}$ form a local basis in $T_{\mathcal{O}}$, this defines a map of \mathcal{O}_P modules, $s : T_{\mathcal{O}} \rightarrow F_{\mathcal{O}}$. One has $[X_{f_i}, X_{f_j}] = 0$ and also $\{f_i, f_j\} = 0$, which proves that s is a Lie algebra morphism. \square

Proposition 3.8. *Let (M, ω, \mathcal{P}) and $(M, \omega', \mathcal{P}')$ be two deformations of (M, ω_0, P) . Then any point of M has an open neighborhood, U , such that there exists a formal automorphism, B , of U identical modulo t that transforms (U, ω, \mathcal{P}) to $(U, \omega', \mathcal{P}')$.*

Proof. For any point of $m \in M$ there is an open neighborhood, V , such that on it $\omega' = \omega + td\gamma$, where γ is an 1-form. Let X_γ be the vector field on V corresponding to γ by the isomorphism $T_M^* \rightarrow T_M$ determined by ω . Then, $\exp(tX_\gamma)$ is a formal automorphism of V which transforms ω to ω' . Hence, one may suppose in the proposition that $\omega = \omega'$ and \mathcal{P} and \mathcal{P}' are two polarizations of the same form ω .

Let U be a neighborhood of m such that over U the Dolbeault complexes corresponding to both \mathcal{P} and \mathcal{P}' are exact. Let functions a_i, a'_i on U be such that da_i and da'_i form local bases in \mathcal{P}^\perp and \mathcal{P}'^\perp , respectively. Suppose $\mathcal{P} = \mathcal{P}' \bmod t^k$. Then, we may assume that $a_i = a'_i \bmod t^k$, i.e. $a'_i = a_i + t^k b_i$ for some functions b_i on U . Let $\{\cdot, \cdot\}$ denote the Poisson bracket inverse to ω . One has $\{a_i, a_j\} = \{a'_i, a'_j\} = 0$, hence $\{a_i, b_j\} + \{b_i, a_j\} = 0$. It follows from the exactness of Dolbeault complex that there exists a function g on U such that $\{a_i, g\} = b_i$ for all i . Let X_g be the Hamiltonian vector field on U corresponding to g . Then, the automorphism $\exp(tX_g)$ of U leaves ω on the place and transforms \mathcal{P} to a polarization, \mathcal{P}'' , such that $\mathcal{P}'' = \mathcal{P}' \bmod t^{k+1}$. Proceeding by iteration proves the proposition. \square

This proposition shows that any deformation (M, ω, \mathcal{P}) of (M, ω_0, P) is, locally, isomorphic to the trivial deformation $(M, \omega_0, P[[t]])$.

3.3 Local properties of polarized star-products

Let (M, ω_0) be a symplectic manifold with a strong polarization P . Let ω be a symplectic deformation of ω_0 and \mathcal{P} a polarization of ω which is a deformation of P . As is proven in the previous section, \mathcal{P} is strong and thus a good polarization. Let \mathcal{O} be the sheaf of functions constant along \mathcal{P} and $(\mathcal{A}, \mathcal{O})$ a polarized star-product constructed in Section 2.

Our immediate goal is to prove analogs of Lemmas 3.4 and 3.6 for the commutator $\frac{1}{t}[\cdot, \cdot]$ in \mathcal{A} .

Lemma 3.9. *Let \mathcal{P} be a strong polarization of ω . Let $a_1, \dots, a_n \in \mathcal{O}$ be functions on an open set $U \subset M$ such that da_1, \dots, da_n form a local basis in \mathcal{P}^\perp . Then, locally, there exist functions $\hat{f}_1, \dots, \hat{f}_n$ such that $\frac{1}{t}[\hat{f}_i, a_j] = \delta_{ij}$.*

Proof. The operators $\frac{1}{t}[a_i, \cdot]$ are pairwise commuting derivations of \mathcal{A} restricted to an open set U . Let L be the free $C_M^\infty[[t]]$ module over U spanned on $[a_i, \cdot]$. Denote by $\hat{a}^i \in L^*$

the basis dual to $\frac{1}{t}[a_i, \cdot]$. Let us define in the obvious way the complex $(\bigwedge_{C_M^\infty[[t]]}^\bullet L^*, \hat{d})$, where

$$\hat{d} = \sum_i [a_i, \cdot] \hat{d}^i. \quad (31)$$

Since $\frac{1}{t}[a_i, \cdot] = \{a_i, \cdot\} + o(t)$ and $\text{rank} L = \text{rank} P$, this complex is a deformation of the Dolbeault complex $(\bigwedge_{C_M^\infty[[t]]}^\bullet P^*, \bar{d})$ over U with \bar{d} defined by (29). Since \mathcal{P} is strong, the Dolbeault complex is exact, therefore the complex $(\bigwedge_{C_M^\infty[[t]]}^\bullet L^*, \hat{d})$ is exact, too.

Now we proceed as in the proof of Lemma 3.4. Namely, for any i , consider the closed form \hat{d}^i . Since the complex $(\bigwedge_{C_M^\infty[[t]]}^\bullet L^*, \hat{d})$ is exact, there is a function \hat{f}_i such that $\hat{d}(\hat{f}_i) = -\hat{d}^i$. By (31), f_i satisfies the lemma. \square

Let

$$F(\mathcal{A}) = \{b \in C_M^\infty[[t]]; [b, \mathcal{O}] \subset \mathcal{O}\}. \quad (32)$$

It is clear that $F(\mathcal{A})$ is a sheaf of Lie algebras on M with the bracket $\frac{1}{t}[\cdot, \cdot]$. It is a locally free \mathcal{O} module with the local basis consisting of functions $1, \hat{f}_j$. \mathcal{O} sits in $F(\mathcal{A})$ as a commutative Lie subalgebra. The sheaf of Lie algebras $F(\mathcal{A})/\mathcal{O}$ is also a locally free \mathcal{O} -module isomorphic to the sheaf $T_{\mathcal{O}} = F_{\mathcal{O}}/\mathcal{O}$ from the previous section: locally, the isomorphism is given by $\frac{1}{t}[\hat{f}, \cdot] \mapsto \{f_i, \cdot\}$.

Lemma 3.10. *By hypothesis of Lemma 3.9, \hat{f}_i can be chosen in such a way that $\frac{1}{t}[\hat{f}_i, \hat{f}_j] = 0$.*

Proof. The same as of Lemma 3.6. \square

Lemma 3.10, in particular, means that locally there exists a “complement” commutative subalgebra in \mathcal{A} . This subalgebra is generated by functions \hat{f}_i . But that “complement” subalgebra may *not* be an algebra of functions constant on a polarization.

Proposition 3.11. *Let \mathcal{P} be a strong polarization of ω and $(\mathcal{A}, \mathcal{O})$ the corresponding polarized quantization. Then the sequence*

$$0 \longrightarrow \mathcal{O} \longrightarrow F(\mathcal{A}) \xrightarrow{\pi} T_{\mathcal{O}} \longrightarrow 0 \quad (33)$$

locally splits as an exact sequence of \mathcal{O} modules and Lie algebras.

Proof. The same as of Proposition 3.7. \square

Proposition 3.12. *Let (M, ω, \mathcal{P}) , $(M, \omega', \mathcal{P}')$ be two deformations of (M, ω_0, P) and $(\mathcal{A}, \mathcal{O})$, $(\mathcal{A}', \mathcal{O}')$ corresponding polarized star-products. Then any point of M has an open neighborhood, U , such that over U these star-products are equivalent. Moreover, if $\mathcal{O} = \mathcal{O}'$ then the equivalence can be chosen to be identity on \mathcal{O} .*

Proof. By Proposition 3.8 one can suppose that $\omega = \omega' = \omega_0$ and $\mathcal{P} = \mathcal{P}' = P[[t]]$. So, $\mathcal{O} = \mathcal{O}' = \mathcal{O}_0[[t]]$, where \mathcal{O}_0 is the algebra of functions constant along P .

Let $m \in M$ and U be a sufficiently small neighborhood of m . Let a_i be functions on U such that da_i form a local basis in P^\perp and f_i be functions such that $\{f_i, a_j\} = \delta_{ij}$, where $\{\cdot, \cdot\}$ is the Poisson bracket inverse to ω_0 . Let $\mu = \sum_{i \geq 0} t^i \mu_i$ and $\mu' = \sum_{i \geq 0} t^i \mu'_i$ be multiplications in $C_M^\infty[[t]]$ over U corresponding to \mathcal{A} and \mathcal{A}' and consisting of bidifferential operators. So, $\mu_0(a, b) = \mu'_0(a, b) = ab$ and one may suppose that $\mu_1(a, b) = \mu'_1(a, b) = \frac{1}{2}\{a, b\}$.

Suppose that there exists a differential operator $B^{(n)} = 1 + \dots + t^n B_n$ which transforms μ to μ' modulo t^{n+1} and such that B_k , $k = 1, \dots, n$, take \mathcal{O} to zero. This means, in particular, that we may suppose that $\mu = \mu'$ modulo t^{n+1} , so that

$$\mu = \mu_0 + \dots + t^n \mu_n + \mu_{n+1} + \dots,$$

$$\mu' = \mu_0 + \dots + t^n \mu_n + \mu'_{n+1} + \dots.$$

We are going to prove that there exists a differential operator of the form $B = 1 + t^n X + t^{n+1} B_{n+1}$, where X is a vector field on U , which transforms μ' to μ modulo t^{n+2} and such that $X(\mathcal{O}) = B_{n+1}(\mathcal{O}) = 0$.

It is easy to check that $c = \mu'_{n+1} - \mu_{n+1}$ is a Hochschild cocycle in the algebra C_U^∞ . Hence, $\nu(a, b) = c(a, b) - c(b, a)$ is a bivector field and there is the representation: $\mu'_{n+1} = \mu_{n+1} + \nu + \delta_H(B'_{n+1})$, where B_{n+1} is a differential operator and δ_H the Hochschild differential.

It is also easy to check that $[\mu_1, \nu] = 0$, where $[\cdot, \cdot]$ denotes the Schouten bracket of polyvector fields. So, there exists a vector field Y such that $[\mu_1, Y] = \nu$. This means that for any $f, g \in C_U^\infty$

$$Y(\mu_1(f, g)) - \mu_1(Yf, g) - \mu_1(f, Yg) = \nu(f, g).$$

Recall now that $\mu_1(a, b) = \nu(a, b) = 0$ for $a, b \in \mathcal{O}$. It follows that for any $a, b \in \mathcal{O}$

$$\mu_1(Ya, b) + \mu_1(a, Yb) = 0.$$

In particular, we have $\mu_1(Ya_i, a_j) + \mu_1(a_i, Ya_j) = 0$ for any a_i, a_j of our basis. It follows from the exactness of Dolbeault complex that there exists a function $b \in C_U^\infty$ such that $Ya_i = \mu_1(b, a_i)$ for all i . Denote $X_b = \{b, \cdot\}$, the Hamiltonian vector field corresponding to b . Put $X = Y - X_b$. Then, since $[\mu_1, X_b] = 0$, we get $[\mu_1, X] = \nu$. Since $Ya_i = X_b a_i$ for all i , we get $X(\mathcal{O}) = 0$.

If we transform the multiplication μ' by the operator $1 + t^n X$, we obtain the multiplication μ'' of the form $\mu'' = \mu_0 + \dots + t^n \mu_n + \mu''_{n+1} + \dots$, where $\mu''_{n+1} = \mu_{n+1} + \delta_H B'_{n+1}$.

Since $\mu''_{n+1}(a, b) = \mu_{n+1}(a, b) = 0$ for all $a, b \in \mathcal{O}$, we have that B'_{n+1} is a derivation from \mathcal{O}_U to C_U^∞ . This derivation can be extended to a derivation of C_U^∞ . Indeed, put $B'_{n+1}(a_i) = g_i$. Then the operator $D = g_i \{f_i, \cdot\}$ is such an extension. Now put $B_{n+1} = B'_{n+1} - D$. One has $\delta_H B_{n+1} = \delta_H B'_{n+1}$. It is easy to see that the operator $1 + t^{n+1} B_{n+1}$ transforms μ'' to μ modulo t^{n+2} . It follows that the operator $(1 + t^{n+1} B_{n+1})(1 + t^n X)$ transforms μ' to μ modulo t^{n+2} . \square

Let (M, ω_0, P) be a polarized symplectic manifold with P a strong polarization. Let U be a sufficiently small neighborhood of a point of M . Let a_i be functions on U such that da_i form a local basis in P^\perp and f_i be functions such that $\{f_i, a_j\} = \delta_{ij}$, where $\{\cdot, \cdot\}$ is the Poisson bracket inverse to ω_0 . Then the bivector field

$$\pi = \sum_i X_{a_i} \wedge X_{f_i}$$

represent this Poisson bracket. Since all X_{a_i} and X_{f_i} pairwise commute, one can construct the Moyal star-product of the form

$$f \star g = \mu_0 \exp\left(\frac{t}{2} X_{a_i} \wedge X_{f_i}\right)(f \otimes g), \quad (34)$$

where μ_0 is the usual multiplication. This star-product is obviously polarized, and we call it the *polarized Moyal star-product*.

Corollary 3.13. *Any polarized star-product is locally equivalent to a polarized Moyal star-product.*

Proof. Follows from the previous proposition. \square

4 A formula for polarized quantization

4.1 \mathcal{O} -extension of $T_{\mathcal{O}}$

The sequences (30) and (33) are examples of an \mathcal{O} -extension of $T_{\mathcal{O}}$. In the following we suppose that \mathcal{O} is the sheaf of functions constant along a strong polarization and $T_{\mathcal{O}} = \text{Hom}_{\mathcal{O}}(\Omega_{\mathcal{O}}^1, \mathcal{O})$. As we have seen, $T_{\mathcal{O}}$ is a locally free \mathcal{O} module of derivations of \mathcal{O} .

Following [BK], [BB], we give the following

Definition 4.1. *An \mathcal{O} -extension of $T_{\mathcal{O}}$ is a locally free sheaf \tilde{T} of \mathcal{O} -modules equipped with a structure of a sheaf of Lie algebras over $\mathbb{C}[[t]]$ (with the bilinear operation denoted, as usual, by $[\cdot, \cdot]$), a section \mathbf{c} of the center of \tilde{T} , and a surjective \mathcal{O} -linear map $\sigma : \tilde{T} \rightarrow T_{\mathcal{O}}$ (the anchor map) which is a Lie algebra homomorphism, whose kernel (which is a sheaf of \mathcal{O} -Lie algebras) is isomorphic to \mathcal{O} (the latter equipped with the trivial Lie bracket). In addition the Leibnitz rule holds: for $f \in \mathcal{O}$, $\tau_1, \tau_2 \in \tilde{T}$, $[\tau_1, f\tau_2] = f[\tau_1, \tau_2] + \sigma(\tau_1)(f)\tau_2$. Thus, there is an exact sequence (of \mathcal{O} -modules and Lie algebras)*

$$0 \longrightarrow \mathcal{O} \xrightarrow{i} \tilde{T} \xrightarrow{\sigma} T_{\mathcal{O}} \longrightarrow 0, \quad (35)$$

where $i(f) = f \cdot \mathbf{c}$, $f \in \mathcal{O}$.

Just as in Proposition 3.7, one shows that the extension (35) admits local splittings which are \mathcal{O} -linear Lie algebra homomorphisms. Such a splitting is called a *flat connection on \tilde{T}* .

4.2 Characteristic class of an \mathcal{O} -extension of $T_{\mathcal{O}}$

To each \tilde{T} one associates the cohomology class $c(\tilde{T}) \in H^1(M; \Omega_{\mathcal{O}}^{1,cl}) = H^2(M, \mathbb{C}[[t]])$ as follows.

Let $\{U_{\alpha}\}$ be a sufficiently fine open covering of M , so that there are flat connections s_{α} on $\tilde{T}|_{U_{\alpha}}$. If $U_{\alpha\beta} \stackrel{def}{=} U_{\alpha} \cap U_{\beta} \neq \emptyset$, the difference $s_{\alpha} - s_{\beta}$ gives rise to the section $c_{\alpha,\beta}$ (defined over $U_{\alpha\beta}$) of $\Omega_{\mathcal{O}}^{1,cl} = \underline{\text{Hom}}_{\mathcal{O}}(T_{\mathcal{O}}, \mathcal{O})$ by the formula $i(c_{\alpha,\beta}(\xi)) = s_{\alpha}(\xi) - s_{\beta}(\xi)$. The collection $\{c_{\alpha,\beta}\}$ is a degree one Čech cochain with coefficients in the sheaf $\Omega_{\mathcal{O}}^{1,cl}$. It is, in fact, a cocycle, whose cohomology class, denoted $c(\tilde{T})$ is independent of the choice of local flat connections.

The above construction recovers the Chern class of a complex line bundle. Namely, suppose that L is a locally free sheaf of \mathcal{O} -modules of rank one. Let $\tilde{T}_{\mathcal{L}}$ denote the sheaf of differential operators on L of order (at most) one. The sheaf $\tilde{T}_{\mathcal{L}}$ is equipped with the *left* \mathcal{O} -module structure, the Lie bracket given by the commutator, the central section – the identity operator (so that the map i is simply the inclusion of operators of order zero). The principal symbol map serves as the anchor map σ . These data exhibit $\tilde{T}_{\mathcal{L}}$ as an \mathcal{O} -extension of $T_{\mathcal{O}}$.

On the other hand, the sheaf $\mathcal{L} \otimes_{\mathcal{O}} C_M^{\infty}$ is the sheaf of C^{∞} sections of a complex line bundle L . It is easy to see that the class $c(\tilde{T}_{\mathcal{L}})$ coincides with the the first Chern class of L .

For example, since $\det \mathcal{P}^{\perp} = \Omega_{\mathcal{O}}^n \otimes_{\mathcal{O}} C_M^{\infty}$, we find that $c_1(\mathcal{P}^{\perp}) = c(\tilde{T}_{\Omega_{\mathcal{O}}^n})$.

For a PDQ (polarized deformation quantization) $(\mathcal{A}, \mathcal{O})$ we set $\text{cl}(\mathcal{A}, \mathcal{O}) = c(F(\mathcal{A}))$.

4.3 The opposite PDQ

For a ring R we denote by R^{op} the opposite ring. That is, R^{op} is a ring, there is a bijection $(\bullet)^{\dagger} : R \rightarrow R^{op}$ such that $1^{\dagger} = 1$ and $r_1^{\dagger} r_2^{\dagger} = (r_2 r_1)^{\dagger}$. We will usually identify R and R^{op} using the bijection.

If \mathcal{A} is a deformation quantization of (M, ω_0) , then, clearly, \mathcal{A}^{op} is a deformation quantization of $(M, -\omega_0)$. If $(\mathcal{A}, \mathcal{O})$ is a PDQ, then $(\mathcal{A}^{op}, \mathcal{O})$ is a PDQ (since $(\bullet)^{\dagger}$ restricts to a ring isomorphism on \mathcal{O}). It is clear that $F(\mathcal{A})$ and $F(\mathcal{A}^{op})$ coincide as subsheaves of \mathcal{A} . However, the respective \mathcal{O} -module structures, symbol maps, and Lie brackets are different. In fact, one has:

- $f\tau^{\dagger} = (f\tau - t\sigma(\tau)(f))^{\dagger} = (f\tau)^{\dagger} - t\sigma(\tau)(f)$
- $\sigma(\tau^{\dagger}) = -\sigma(\tau)$
- $[\tau_1^{\dagger}, \tau_2^{\dagger}] = -[\tau_1, \tau_2]^{\dagger}$.

Proposition 4.2. *Suppose that $(\mathcal{A}, \mathcal{O})$ is a PDQ. Then,*

$$\text{cl}(\mathcal{A}^{op}, \mathcal{O}) = c_1(\mathcal{P}^{\perp}) - \text{cl}(\mathcal{A}, \mathcal{O})$$

Proof. Choose an open covering $\{U_{\alpha}\}$ of M such that, for each open set U_{α} there exist flat connections s_{α} on $F(\mathcal{A})|_{U_{\alpha}}$ and s_{α}^{Ω} on $\tilde{T}_{\Omega_{\mathcal{O}}^n}|_{U_{\alpha}}$.

Let ℓ denote the splitting of the extension

$$0 \longrightarrow \mathcal{O} \longrightarrow \tilde{T}_{\Omega_{\mathcal{O}}} \longrightarrow T_{\mathcal{O}} \xrightarrow{\sigma} 0$$

given by the *negative* of the Lie derivative. This splitting is a Lie algebra homomorphism, but is *not* \mathcal{O} -linear. Instead, for $f \in \mathcal{O}$ and $\xi \in T$, $\ell(f\xi) = f\ell(\xi) + \xi(f)$. We denote the restriction of this splitting to U_{α} by the same letter.

Since both ℓ and s_{α}^{Ω} are splittings of the same sequence, their difference $\psi_{\alpha} = s_{\alpha}^{\Omega} - \ell$ is a map $\psi_{\alpha} : T \rightarrow \mathcal{O}$ which is *not* \mathcal{O} -linear, but satisfies $\psi_{\alpha}(f\xi) = f\psi_{\alpha}(\xi) + \xi(f)$. Note that $\psi_{\alpha} - \psi_{\beta} = s_{\alpha}^{\Omega} - s_{\beta}^{\Omega}$.

Let $s_{\alpha}^{op}(\xi) = t\psi_{\alpha}(\xi) - s_{\alpha}(\xi)^{\dagger}$. We claim that s_{α}^{op} is a (locally defined) flat connection on $F(\mathcal{A}^{op})$. To this end we check various properties.

The calculation

$$\sigma(s_{\alpha}^{op}(\xi)) = \sigma(t\psi_{\alpha}(\xi) - s_{\alpha}(\xi)^{\dagger}) = \sigma(-s_{\alpha}(\xi)^{\dagger}) = \sigma(s_{\alpha}(\xi)) = \xi$$

shows that s_{α}^{op} is a (local) splitting. The calculation

$$\begin{aligned} s_{\alpha}^{op}(f\xi) &= t\psi_{\alpha}(f\xi) - s_{\alpha}(f\xi)^{\dagger} = \psi_{\alpha}(f\xi) - (fs_{\alpha}(\xi))^{\dagger} = tf\psi_{\alpha}(\xi) + t\xi(f) - (fs_{\alpha}(\xi))^{\dagger} + t\xi(f) = \\ &tf\psi_{\alpha}(\xi) - fs_{\alpha}(\xi)^{\dagger} = fs_{\alpha}^{op}(\xi) \end{aligned}$$

shows that this splitting is \mathcal{O} -linear.

One verifies other properties similarly.

Hence one can use the locally defined flat connections $\{s_{\alpha}^{op}\}$ to calculate the characteristic class of $F(\mathcal{A}^{op})$. The desired formula follows from

$$s_{\alpha}^{op} - s_{\beta}^{op} = (\psi_{\alpha}(\xi) - s_{\alpha}(\xi)) - (\psi_{\beta}(\xi) - s_{\beta}(\xi)) = (s_{\alpha}^{\Omega} - s_{\beta}^{\Omega}) - (s_{\alpha} - s_{\beta})$$

□

4.4 $\text{cl}(\mathcal{A}, \mathcal{O})$ and other classes

Let \mathcal{A}_1 and \mathcal{A}_2 be two deformation quantizations on (M, ω_0) . Deligne, [De], related to them cohomology classes

$$\begin{aligned} [\mathcal{A}_2 : \mathcal{A}_1] &\in H^2(M, \mathbb{C}[[t]]) \quad \text{and} \\ \text{obs}_{DL}(\mathcal{A}_1) &\in \frac{1}{t}[\omega_0] + H^2(M, \mathbb{C}[[t]]). \end{aligned}$$

He proved that

$$[\mathcal{A}_2 : \mathcal{A}_1] = \theta(\mathcal{A}_2) - \theta(\mathcal{A}_1) \quad \text{and} \quad (36)$$

$$\text{obs}_{DL}(\mathcal{A}_1) = -t \frac{d}{dt} \theta(\mathcal{A}_1). \quad (37)$$

Lemma 4.3. *Let $(\mathcal{A}_1, \mathcal{O}_1)$ and $(\mathcal{A}_2, \mathcal{O}_2)$ be two polarized quantizations of (M, ω_0, P) . Then $[\mathcal{A}_2 : \mathcal{A}_1] = \frac{1}{t} \text{cl}(\mathcal{A}_2, \mathcal{O}_2) - \frac{1}{t} \text{cl}(\mathcal{A}_1, \mathcal{O}_1)$.*

Proof. By Proposition 3.12, there is an open covering $\{U_\alpha\}$ of M such that there exist isomorphisms of polarized quantizations

$$B_\alpha : (\mathcal{A}_1, \mathcal{O}_1) |_{U_\alpha} \rightarrow (\mathcal{A}_2, \mathcal{O}_2) |_{U_\alpha}.$$

These isomorphisms induce isomorphisms $F(\mathcal{A}_1) \rightarrow F(\mathcal{A}_2)$ over each U_α . We may suppose that both $F(\mathcal{A}_1)$ and $F(\mathcal{A}_2)$ admit on each U_α flat connections $s_{i\alpha} : T_{\mathcal{O}_i} \rightarrow F(\mathcal{A}_i)$, $i = 1, 2$, such that $B_\alpha(s_{1\alpha}) = s_{2\alpha}$. Since $F(\mathcal{A}_1) = F(\mathcal{A}_2) \bmod t$, we may choose the connections in such a way that $s_{1\alpha} = s_{2\alpha} \bmod t$.

Let $f_{i\alpha\beta} \in \mathcal{O}_i$ be functions on $U_\alpha \cap U_\beta$ such that $df_{i\alpha\beta} = s_{i\beta} - s_{i\alpha}$. Since $s_{1\alpha} = s_{2\alpha} \bmod t$, we may choose $f_{i\alpha\beta}$ such that $f_{1\alpha\beta} = f_{2\alpha\beta} \bmod t$. On $U_\alpha \cap U_\beta$ one has $B_\alpha^{-1}B_\beta = \exp(\frac{1}{t} \text{ad}(f_{2\alpha\beta} - f_{1\alpha\beta}))$. Since $f_{2\alpha\beta} - f_{1\alpha\beta} \in t\mathcal{O}_i$, $\exp(\frac{1}{t} \text{ad}(f_{2\alpha\beta} - f_{1\alpha\beta}))$ is an automorphism of \mathcal{A}_1 on $U_\alpha \cap U_\beta$. According to Deligne's definition, $[\mathcal{A}_2 : \mathcal{A}_1]$ is equal to $\frac{1}{t} \tilde{\partial}(f_{2\alpha\beta} - f_{1\alpha\beta})$. \square

Lemma 4.4. *Let $(\mathcal{A}, \mathcal{O})$ be a polarized quantization of (M, ω_0, P) corresponding to a deformation (M, ω, \mathcal{P}) . Then, $\text{obs}_{DL}(\mathcal{A}) = -t \frac{d}{dt}(\frac{1}{t} \text{cl}(\mathcal{A}, \mathcal{O}))$.*

Proof. First, we assume that $(\mathcal{A}, \mathcal{O})$ corresponds to the trivial deformation $(M, \omega_0, P[[t]])$.

By Proposition 3.12, there is an open covering $\{U_\alpha\}$ of M such that $(\mathcal{A}, \mathcal{O}) |_{U_\alpha}$ are isomorphic to Moyal polarized quantizations. Let D_α be \mathbb{C} linear derivations of $\mathcal{A} |_{U_\alpha}$ such that for $a \in \mathcal{O}$ one has $D_\alpha a = t \frac{d}{dt} a$. Such D_α exist because they obviously exist for the Moyal star-products. Then, $D_\alpha F(\mathcal{A}) \subset F(\mathcal{A})$.

Let the covering $\{U_\alpha\}$ be fine enough and on each U_α there exists a flat connection $s_\alpha : T_{\mathcal{O}} \rightarrow F(\mathcal{A})$. Then, on $U_\alpha \cap U_\beta$ one has $s_\beta - s_\alpha = df_{\alpha\beta}$ for $f_{\alpha\beta} \in \mathcal{O}$. Let $a_{\alpha 1}, \dots, a_{\alpha n}$ be functions not depending on t such that $da_{\alpha 1}, \dots, da_{\alpha n}$ form a local basis in P^\perp , and $\partial_{\alpha 1}, \dots, \partial_{\alpha n}$ be the dual basis in $T_{\mathcal{O}}$. Put $f_{\alpha i} = s_{\alpha i}(\partial_{\alpha i})$.

One has $[f_{\alpha i}, a_{\alpha j}] = t\delta_{ij}$. Hence, since $a_{\alpha i}$ does not depend of t , $[D_\alpha f_{\alpha i}, a_{\alpha j}] = t\delta_{ij}$. This means that $D_\alpha f_{\alpha i} = f_{\alpha i} + b_{\alpha i}$ for some $b_{\alpha i} \in \mathcal{O}$. It follows from $[f_{\alpha i}, f_{\alpha j}] = 0$ that $[f_{\alpha i}, b_{\alpha j}] + [b_{\alpha i}, f_{\alpha j}] = 0$. It follows from the exactness of \mathcal{O} -de Rham complex that there exist $b_\alpha \in \mathcal{O}$ such that $b_{\alpha i} = \frac{1}{t} [b_\alpha, f_{\alpha i}]$. So, $D_\alpha f_{\alpha i} = f_{\alpha i} + \frac{1}{t} [b_\alpha, f_{\alpha i}]$. Taking $D_\alpha - \frac{1}{t} \text{ad}(b_\alpha)$ instead D_α we obtain that the new D_α satisfy $D_\alpha f_{\alpha i} = f_{\alpha i}$. This equations may be rewritten in the form

$$D_\alpha s_\alpha = s_\alpha. \tag{38}$$

Further, one has

$$s_\alpha = s_\beta - [\frac{1}{t} f_{\alpha\beta}, s_\beta]. \tag{39}$$

Using (38) and (39), we have

$$(D_\beta - D_\alpha)s_\alpha = D_\beta(s_\beta - [\frac{1}{t} f_{\alpha\beta}, s_\beta]) - D_\alpha s_\alpha = -[D_\beta(\frac{1}{t} f_{\alpha\beta}), s_\beta] = [-t \frac{d}{dt}(\frac{1}{t} f_{\alpha\beta}), s_\beta].$$

Since derivations of \mathcal{A} are completely defined by their values on $F(\mathcal{A})$, it follows that $D_\beta - D_\alpha = \text{ad}(-t \frac{d}{dt}(\frac{1}{t} f_{\alpha\beta}))$. According to Deligne's definition, this means that $(-t \frac{d}{dt}(\frac{1}{t} f_{\alpha\beta}))$ represents $\text{obs}_{DL}(\mathcal{A})$.

Now, let (M, ω, \mathcal{P}) be arbitrary. Denote by cl the class of a quantization corresponding to (M, ω, \mathcal{P}) and by cl_0 the class of a quantization corresponding to the trivial deformation $(M, \omega_0, P[[t]])$. The same meaning has the notation θ , θ_0 and obs_{DL} , obs_{DL0} .

By Lemma 4.3 and (36) we have $\frac{1}{t}\text{cl}_0 - \frac{1}{t}\text{cl} = \theta_0 - \theta$, hence

$$\theta_0 - \frac{1}{t}\text{cl}_0 = \theta - \frac{1}{t}\text{cl}.$$

Applying $-t \frac{d}{dt}$ to the left hand side of this equation, we obtain zero (by (37) and what we have just proved). Hence, applying $t \frac{d}{dt}$ to the right hand side and using (37), we have

$$\text{obs}_{DL} = -t \frac{d}{dt} \theta = -t \frac{d}{dt} \text{cl},$$

which proves the proposition. \square

It follows from Lemma 4.3 that the difference $\theta(\mathcal{A}) - \frac{1}{t}\text{cl}(\mathcal{A}, \mathcal{O})$ does not depend on the quantization $(\mathcal{A}, \mathcal{O})$ of (M, ω_0, P) , while Lemma 4.4 shows that this difference does not depend on t , i.e.

$$\theta(\mathcal{A}) = \frac{1}{t}\text{cl}(\mathcal{A}, \mathcal{O}) - c, \tag{40}$$

where $c \in \mathbb{C}$.

4.5 Relation between $\theta(\mathcal{A})$ and $\text{cl}(\mathcal{A}, \mathcal{O})$

In this Section we calculate the constant c in (40).

Let \mathcal{A} be a deformation quantization of (M, ω_0) . Let \mathcal{A}^{op} denote the opposite algebra and \mathcal{A}^σ denote the algebra obtained by the automorphism of $\mathbb{C}[[t]]$ taking $t \mapsto -t$. It is clear that both \mathcal{A}^{op} and \mathcal{A}^σ are quantizations of $(M, -\omega_0)$.

Lemma 4.5. *Let \mathcal{A} be a deformation quantization on (M, ω_0) . Then*

- a) $\theta(\mathcal{A}^{op}) = -\theta(\mathcal{A})$;
- b) $\theta(\mathcal{A}^\sigma) = \theta(\mathcal{A})$.

Proof. Follows from the Fedosov construction, see also [De]. \square

Theorem 4.6. *Let $(\mathcal{A}, \mathcal{O})$ be a polarized quantization of (M, ω_0, P) . Then*

$$\theta(\mathcal{A}) = \frac{1}{t}\text{cl}(\mathcal{A}, \mathcal{O}) - \frac{1}{2}c_1(P). \tag{41}$$

Proof. By Lemma 4.3 we have the equalities

$$\theta(\mathcal{A}^{op}) - \theta(\mathcal{A}^\sigma) = [\mathcal{A}^{op} : \mathcal{A}^\sigma] = \text{cl}(\mathcal{A}^{op}, \mathcal{O}) - \text{cl}((\mathcal{A}, \mathcal{O})^\sigma) .$$

By Lemma 4.5 and Proposition 4.2 we have $\theta(\mathcal{A}^{op}) = -\theta(\mathcal{A})$ and $\text{cl}(\mathcal{A}^{op}, \mathcal{O}) = -\text{cl}(\mathcal{A}, \mathcal{O}) + c_1(\Omega_{\mathcal{O}}^1)$. Combining these with the preceding identity we obtain

$$(\text{cl}(\mathcal{A}, \mathcal{O}) - \theta(\mathcal{A})) + (\text{cl}((\mathcal{A}, \mathcal{O})^\sigma) - \theta(\mathcal{A}^\sigma)) = c_1(\Omega_{\mathcal{O}}^1) .$$

By (40) both differences in the left hand side of the last identity are equal to the differences of the respective constant terms, hence coincide (since the constant terms are not affected by σ). Thus the above expression says

$$2(\text{cl}(\mathcal{A}, \mathcal{O}) - \theta(\mathcal{A})) = c_1(\Omega_{\mathcal{O}}^1) .$$

The theorem follows from the fact that $c_1(\Omega_{\mathcal{O}}^1) = c_1(P)$. \square

Note that the formula (41) relates to the formula (23), which shows that $\text{cl}(\mathcal{A}, \mathcal{O}) = \frac{1}{t}[\Omega_D]$, where Ω_D is the Wick curvature appearing in the Fedosov procedure of constructing $(\mathcal{A}, \mathcal{O})$.

References

- [BB] Beilinson A. and Bernstein J., A proof of Jantzen conjectures. I. M. Gelfand Seminar, 1–50, Adv. Soviet Math., 16, Part 1, Amer. Math. Soc., Providence, RI, 1993.
- [BK] Beilinson A. and Kazhdan D., Geometric quantization, preprint.
- [BW] Bordemann M. and Waldmann S., A Fedosov star product of the Wick type for Kähler manifolds. Lett. Math. Phys. 41 (1997), no. 3, 243–253.
- [De] Deligne P., Dèformations de l’Algèbre des Fonctions d’une Variété Symplectique: Comparaison entre Fedosov et De Wilde, Lecomte, Selecta Mathematica, New Series, v. 1, n. 4 (1995), 667–697.
- [Fe] Fedosov B., “Deformation Quantization and Index Theory”, Akademie Verlag, v. 9.
- [Ka1] Karabegov A., Deformation Quantization with Separation of Variables on Kähler Manifold, Commun. Math. Phys., v. 180 (1996), 745–755.
- [Ka2] Karabegov A., Cohomological Classification of Deformation Quantizations with Separation of Variables, Lett. Math. Phys., v. 43 (1998), 347–357.
- [NT] Nest R., Tsygan B., Algebraic Index Theorem for Families, Adv. Math. 113 (1995), 151–205.
- [Ra] Rawnsley J.H., Polarization and diagonal quantization, Trans. AMS, 230, 1977, 235–255.
- [RY] Reshetikhin N., Yakimov M., Deformation Quantization of Lagrangean Fiber Bundles, preprint, math/9907164.
- [Xu] Xu P., Fedosov’s \star -products and quantum momentum maps, Comm. Math. Phys., 197 (1998), 167–197.

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